Name: ..………………………………………………………………………………….

Student Number: ……………………………………………………………………..

Test 4 on WPPH16001.2019-2020
"Electricity and Magnetism"

Content: 9 pages (including this cover page)

Friday May 1 2020; **online**, 15:00-17:30

- Write your full name and student number on **each** page you use
- Read the questions carefully. Read them one more time after having answered them.
- Compose your answers is such a way that it is well indicated which (sub)question they address
- Upload the answer to each question as a **separate pdf file**
- Do not use a red pen (it's used for grading)
- Griffiths' textbook, lecture notes and **your** tutorial notes are allowed. The internet, mobile phones, consulting and other teamwork are not allowed (and considered as cheating)

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

For administrative purposes; do NOT fill the table

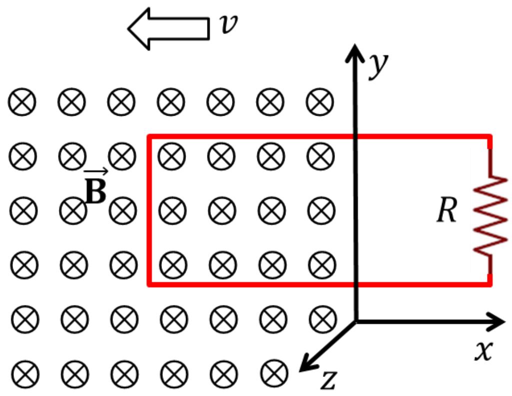
The weighting of the questions:

	Maximum points	Points scored
Question 1	~~15~~ 10	
Question 2	10	
Question 3	15	
Question 4	15	
Total	**~~55~~ 50**	

Grade = 1 + 9 x (score/max score).

Grade: __________

Question 1. ~~(15 points)~~ **(10 points)**



Let's consider the "second" Faraday experiment, this time from the viewpoint of Faraday's equation. In this experiment, Faraday moved the magnetic field (the magnet) to the left, holding the loop still. Suppose that at time $t = 0$ the magnetic field $\vec{\mathbf{B}}$ in a direction into the paper fills the region of $x < 0$. As we know, the current flowed through the (red) loop so that some work was done.

1. Is it the magnetic field which does the work? (1 point)

2. Where is $\frac{\partial \vec{\mathbf{B}}}{\partial t}$ nonzero? Sketch $\frac{\partial \vec{\mathbf{B}}}{\partial t}$ schematically (qualitatively) in the figure. Don't forget about its direction! (5 points)

Tip: approximate graphically the abrupt change in the $\vec{\mathbf{B}}$-field at $x = 0$ (i.e. at the magnet edge) by any smooth function, draw its derivative, and then make the smooth function sharper and sharper.

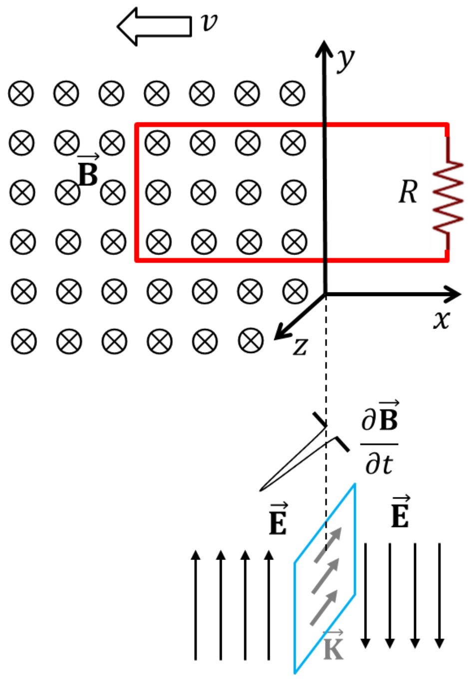
3. Exploit the analogy between Faraday's and Ampère's law to sketch qualitatively the electric field. ~~(5 points)~~ (4 points)

~~4. At which segments of the loop is the work done on electrons? Explain why. (2 points)~~

~~5. Now apply the Lenz rule to establish the current direction. Does the current direction coincide with what you predicted in #3? (2 points)~~

Model answers (Griffiths, Problem 7.20 modified) (15 points)

1. No, magnetic field never works. (1 point)



2. $\frac{\partial \vec{\mathbf{B}}}{\partial t}$ is nonzero along the edge of the magnetic field (1 point)
Direction is $+z$ (1 point).
See the figure for the shape (3 points)
(Strictly speaking, its analytical form is a δ-function $B \cdot \delta(t)$, a derivative of a Heaviside step function).

Note added: in the exam chat, it was allowed to draw the derivative in a separate graph, with the diction in space indicated.

3. The $\vec{\mathbf{E}}$-field resulting from $-\frac{\partial \vec{\mathbf{B}}}{\partial t}$ is analogous to the $\vec{\mathbf{B}}$-field from a current sheet in the yz plane under Ampère's law,

$$-\frac{\partial \vec{\mathbf{B}}}{\partial t} \Leftrightarrow \mu_0 \vec{\boldsymbol{J}}$$

with the (surface) current $\vec{\mathbf{K}}$ running into -z direction (considering the minus sign). (1 point)

We know (see Example 5.8 and Lecture 19), that the magnetic field from a current sheet is directed parallel to the current sheet and its magnitude is constant at any point. (1 point)

Therefore, the electric field in our example runs parallel the magnet edge at $+y$ direction for $x < 0$ (i.e. inside the magnetic field) and -y direction for $x > 0$ (i.e. outside the magnetic field). (3 points)

4. The electric field does the work on the electrons only at the vertical segments of the loop. (1 point)
In other segments, the electrons are confined in the direction of the electric field so that they cannot move into this direction. (1 point)

5. According to the Lenz law, the current induced in the loop should have such a direction, that the magnetic field, created by this current, counteracts the change in the flux. The flux is decreasing as the $\vec{\mathbf{B}}$**-**field is moved to the left; therefore, the magnetic field of the loop should be directed to the same direction as $\vec{\mathbf{B}}$ which necessitates the clockwise current. (1 point)
The current direction coincides with the direction of the electric field in #3, i.e. clockwise. (1 point)

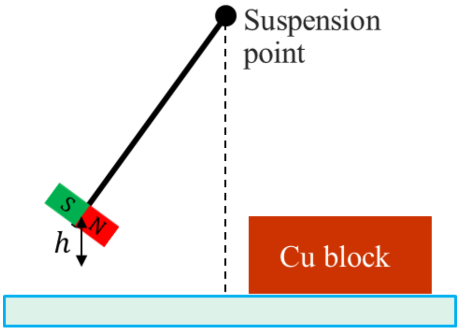
NB: Later in the course we will see how this problem can be elegantly solved (in one line!) by applying Lorentz transformations to the electromagnetic field.

<u>Typical mistakes:</u>
- Not realizing that one can treat **J** similarly to d**B**/dt.
- Not knowing where work is being done on the loop: many say it is done everywhere for there to be a current.
- |d**B**/dt| is nonzero at points where it does not change at all (e.g. the entire loop, everywhere where there is a non-vanishing field, everywhere there is a vanishing field)
- Confusing the direction of motion of the magnet with the direction of the derivative (d**B**/dt points in the direction in which the magnet is moving)
- Drawing the electric field in circles around the vertical segments.

For most of the students, this conceptual problem appears by far the most difficult to solve (however, there were few students offered right answers!). Therefore, I decided to take off subquestions #4 and #5, and decrease the number of points for subquestion #3 to 4 points. This results in the maximum points awarded for this problem to 10, or, equivalently, the maximum number of points was reduced to 50. Who did present correct answers, don't worry: your points remain with you.

Question 2. (10 points)

A pendulum consists of a light rod suspended on a pivot, and a strong neodymium magnet at the lowest end of the rod (see the figure). Near the point of equilibrium, a massive copper block is situated at the trajectory of the pendulum. The pendulum is taken out of the equilibrium position and released.

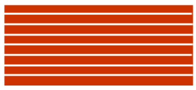


1. Would the speed v at the lowest position be $v^2/2 = gh$ as followed from the conservation of mechanical energy? Explain your answer. (3 points)

2. How does the “lost” energy manifest itself? In other words, to which form of energy is the initial potential energy eventually transformed? (1 point)

3. Now the copper block is cut onto thin vertical slices which are isolated with negligibly thin layers of a dielectric. What changes in the pendulum motion as compared to #1? Explain your answer with a drawing. (2 points)

4. Now the copper block is cut onto thin horizontal slices which are isolated with negligibly thin layers of a dielectric. What changes in the pendulum motion as compared to #1? Explain your answer with a drawing. (2 points)



5. Now the initial (unsliced) copper block is replaced with a similar block made of graphite. What changes in the pendulum motion? Explain your answer. (2 points)

<u>Tip</u>: you might consult Table 7.1 of Griffiths.

<u>Model answers (10 points)</u>

1. As the magnet begins to approach the copper block, the increasing in time flux of the magnetic field induces eddy’s currents in copper. (1 point)

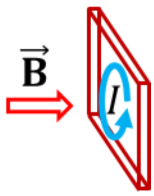
According to the Lenz law, these currents flow in a direction to compensate the change in flux, generating a magnetic field that is opposite in direction to the magnet one. This creates a “drag” force on the magnet. (1 point)

Faster the magnet moves, faster the flux changes, stronger the “drag” force so that the speed prescribed by the conservation of mechanical energy low, will not be reached. (1 point)

In the extreme case, the magnet would stop still near the front edge of the copper block (see https://youtu.be/sENgdSF8ppA).

2. According to Joule’s law, to heat. (1 point)

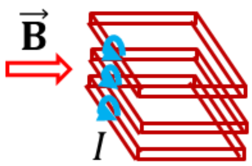
3. In the first approximation, the (changing) magnetic field is orthogonal to the front surface of the copper block. To compensate for the increase of this field, eddy’s currents should have a circular direction in the plane parallel to the from surface. (1 point)



Because copper is non-magnetic, the magnetic field penetrates to the slices beneath the front surface. Therefore, cutting the block in vertical slices does not disturb eddy’s currents too much. Hence, little change in the pendulum motion as compared to the bulk block. (1 point)

4. Now eddy’s currents are cut into smaller pieces by the dialectic which disturbs the circular currents. (1 point)

Hence, smaller deceleration of the pendulum and higher speed at the lower point. (1 point)



5. Conductivity of graphite is a factor of 1000 lower than copper. Therefore, the same emf will create lower eddy's currents. (1 point)
Hence, lower counteracting magnetic field, hence lower the drag force and higher the speed at the lowest point. (1 point)

Typical mistakes:

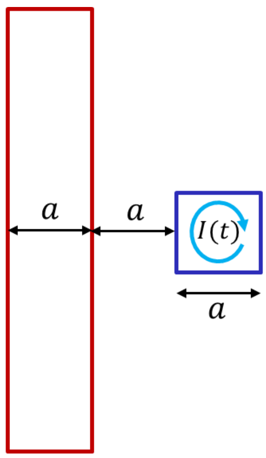
Q2.2: naming the energy in the Eddy currents themselves. Strictly speaking, it is not correct because of "eventually", but we still gave points.

Q2.3: difficulty in visualization the system (even despite the drawing given). The directions of the Eddy currents were drawn in the plane of the paper for a large portion of the exams.

Q2.5: "a higher resistivity leads to more heat and thus more damping" – the reason of damping is eddy's currents and their magnetic field which both are weaker for a higher resistance material.

Question 3. (15 points)

A square loop of wire, of side a, lies at a distance a from one of two long wires, a distance of a apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the small square loop is gradually increasing: $dI/dt = k$ (k is a constant).



1. Find the mutual inductance of the loops. (7 points)
2. Find the emf induced in the big loop. (2 points)
3. What is the direction of the magnetic field produced by the small loop? (2 points)
4. Which way will the induced current flow in the big loop? (2 points)
5. How do the answers change if the separation between the long wires is not a but zero? (2 points)

Model answers (Problem 7.23 modified): (15 points)

1. (7 points)

It is hard to calculate M using a current in the little loop, so, exploiting the *equality of the mutual inductances*, we will find the flux through the little loop when a current I flows in the big loop

$\Phi = MI$ (1 point)

The field of one long wire is $B = \frac{\mu_0 I}{2\pi s}$ so that the flux from the right long wire

$$\Phi_r = \frac{\mu_0 I}{2\pi}\int_a^{2a} \frac{1}{s} a\, ds = \frac{\mu_0 I a}{2\pi}\ln 2$$ (2 points)

The flux from the left long wire

$$\Phi_l = -\frac{\mu_0 I}{2\pi}\int_{2a}^{3a} \frac{1}{s} a\, ds = -\frac{\mu_0 I a}{2\pi}\ln\frac{3}{2}$$ (2 points)

The minus sign is because the direction of the magnetic field is the opposite.

The total flux is

$$\Phi = \Phi_r + \Phi_l = \frac{\mu_0 I a}{2\pi}\ln\frac{4}{3}$$ (1 point)

$$M = \frac{\mu_0 a}{2\pi}\ln\frac{4}{3}$$ (1 point)

2. (2 points)

$$\mathcal{E} = -M\frac{dI}{dt} = -Mk$$ (1 point)

$$\mathcal{E} = -\frac{\mu_0 k a}{2\pi}\ln\frac{4}{3}$$ (1 point)

3. The current in the small loop flows clockwise. This means that the magnetic field of the small loop points into the page inside the loop (1 point)
and out of the page outside the loop (1 point)

4. The magnetic field of the small loop increases with time. So the increase also points out of the page. The current induced in the big loop should be oriented as to oppose this change (Mr Lenz). Therefore, the induced field will point into of the page. Hence, the current in the big loop will be oriented **clockwise**. (2 points)

5. This is equivalent to upwards and downwards currents in the big loop to coincide so that the total current is zero. Therefore, all quantities are equal zero. (2 points)

Typical mistakes:

3.1: Failure to realise that the flux contributions from the left and right wires are different so that the flux from one wire was multiplied by two.

Failure to realise that the fluxes have opposite signs

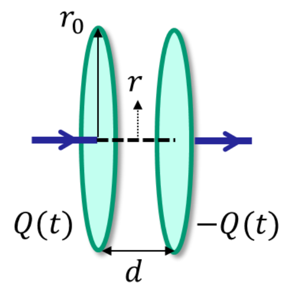
3.3: Forgot to give the direction of the field *outside* the small loop.

3.4: Wrong current direction because of failure in 3.3.

3.5 Failure to realise that the big loop isn't a loop anymore, and no current would flow in it.

Question 4. (15 points)

A battery with potential difference $\mathcal{E}$ charges an ideal circular parallel-plate capacitor of capacitance C, plate radius r_0 and separation between the plates d, through a wire with resistance R. The total charge on each plate as a function of time is: $Q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right)$. Consider the surface charge density uniform on the plates. Find:



1. Displacement current density (3 points)
2. The flux of the electric field (clarification was posted in the chat) between the plates as a function of time (2 points)
3. The rate of change of the flux of the electric field between the plates as a function of time (1 point)
4. The magnetic field inside the capacitor at a distance r from the central axis (5 points)
5. The instantaneous (i.e. at a given moment of time) energy of the magnetic field inside the capacitor (4 points)

Model answers (Problem 7.34 modified, also considered at Lecture 21): (15 points)

1. $\vec{\mathbf{J}}_d \equiv \epsilon_0 \dfrac{\partial \vec{\mathbf{E}}}{\partial t} = \epsilon_0 \dfrac{\partial}{\partial t}\left(\dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}}\right)$ (1 point)

$= \dfrac{\partial}{\partial t}\left(\dfrac{C\mathcal{E}\left(1 - e^{-t/RC}\right)}{\pi r_0^2}\hat{\mathbf{n}}\right) = \dfrac{C\mathcal{E}}{\pi r_0^2}\dfrac{e^{-t/RC}}{RC}\hat{\mathbf{n}}$ (1 point)

$= \dfrac{1}{\pi r_0^2}\dfrac{\mathcal{E}}{R}e^{-t/RC}\,\hat{\mathbf{n}}$ (1 point)

(3 points in total. No vector – minus 1 point)

BTW, here it is apparent that the value is the current *density*. The displacement *current* I_d can be found by integrating the current density over the area πr_0^2: $I_d = \frac{\mathcal{E}}{R}e^{-t/RC}$

2. $\Phi_E = \displaystyle\int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \dfrac{C\mathcal{E}\left(1 - e^{-t/RC}\right)}{\epsilon_0 \pi r_0^2}\int_S da$ (1 point)

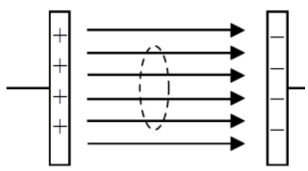
$= \dfrac{C\mathcal{E}}{\epsilon_0}\left(1 - e^{-t/RC}\right)$ (1 point)

(2 points in total)

3. $\dfrac{d\Phi_E}{dt} = \dfrac{C\mathcal{E}}{\epsilon_0 RC}e^{-t/RC} = \dfrac{\mathcal{E}}{\epsilon_0 R}e^{-t/RC}$ (1 point)

4. From the symmetry, the magnetic field has a circumferential direction. (1 point)

We take the circular Amperian loop with radius r which plane is parallel to the plates and which centre coincides with the axis of symmetry, as we did at lectures. (1 point)



Now we use Ampère-Maxwell's law

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{l}} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}}$$

where we have already calculated the derivative at the right-hand part. However, we should recall that not all the flux penetrates the Ampèrian loop but only a fraction r^2/r_0^2. (1 point)

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{l}} = B \cdot 2\pi r = \mu_0 \epsilon_0 \frac{r^2}{r_0^2} \frac{\mathcal{E}}{\epsilon_0 R} e^{-t/RC} \quad \text{(1 point)}$$

$$\vec{\mathbf{B}} = \frac{\mu_0 \mathcal{E}}{2\pi R} e^{-t/RC} \frac{r}{r_0^2} \hat{\boldsymbol{\varphi}}$$ (1 point. Note that the direction could have been defined above)

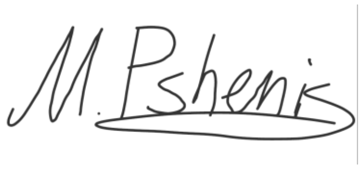
5. $$W_{mag} = \frac{1}{2\mu_0} \int B^2 d\tau = \frac{1}{2\mu_0} \left(\frac{\mu_0 \mathcal{E}}{2\pi R} e^{-t/RC} \frac{1}{r_0^2} \right)^2 \iiint r^2\, r\, dr d\varphi dz \quad \text{(2 points)}$$

$$= \frac{\mu_0}{2} \frac{\mathcal{E}^2}{(2\pi)^2 R^2} \frac{1}{r_0^4} e^{-2t/RC} \frac{r_0^4}{4} 2\pi d \quad \text{(1 point)}$$

$$= \frac{\mu_0}{16} \frac{\mathcal{E}^2 d}{\pi R^2} e^{-2t/RC} \quad \text{(1 point)}$$

Typical mistakes:

- missing direction of vectors -- by far the most common mistake
- not cancelling out the variable “C” when it was present in both the numerator and denominator.
- Introducing variables which were not given (e.g. the area A instead of $\pi {r_0}^2$)
- Failure to take a derivative
- The Jacobian factor missing for integration in cylindrical coordinates and/or ignoring the ‘r’ dependence of B.



Maxim Pchenitchnikov
April 28 2020

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